Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **70**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

Nontrivial solutions for fractional q-difference boundary value problems

Rui A. C. Ferreira*
Department of Mathematics
Lusophone University of Humanities and Technologies
1749-024 Lisbon, Portugal

November 21, 2010

Abstract

In this paper, we investigate the existence of nontrivial solutions to the nonlinear q-fractional boundary value problem

$$(D_q^\alpha y)(x) = -f(x, y(x)), \quad 0 < x < 1,$$
$$y(0) = 0 = y(1),$$

by applying a fixed point theorem in cones.

Keywords: Fractional q-difference equations, boundary value problem, nontrivial solution.

2010 Mathematics Subject Classification: 39A13, 34B18, 34A08.

1 Introduction

The q-difference calculus or *quantum* calculus is an old subject that was first developed by Jackson [9, 10]. It is rich in history and in applications as the reader can confirm in the paper [6].

*Email: ruiacferreira@ulusofona.pt

The origin of the fractional q-difference calculus can be traced back to the works by Al-Salam [3] and Agarwal [1]. More recently, perhaps due to the explosion in research within the fractional calculus setting (see the books [13, 14]), new developments in this theory of fractional q-difference calculus were made, specifically, q-analogues of the integral and differential fractional operators properties such as q-Laplace transform, q-Taylor's formula [4, 15], just to mention some.

To the best of the author knowledge there are no results available in the literature considering the problem of existence of nontrivial solutions for fractional q-difference boundary value problems. As is well-known, the aim of finding nontrivial solutions is of main importance in various fields of science and engineering (see the book [2] and references therein). Therefore, we find it pertinent to investigate on such a demand within this q-fractional setting.

This paper is organized as follows: in Section 2 we introduce some notation and provide to the reader the definitions of the q-fractional integral and differential operators together with some basic properties. Moreover, some new general results within this theory are given. In Section 3 we consider a Dirichlet type boundary value problem. Sufficient conditions for the existence of nontrivial solutions are enunciated.

2 Preliminaries on fractional q-calculus

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^0 = 1, \quad (a-b)^n = \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N}, \quad a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \text{ and } \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.$$

Similarly as done for derivatives, it can be defined an operator I_q^n, namely,

$$(I_q^0 f)(x) = f(x) \text{ and } (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [11]. We point out here four formulas that will be used later, namely, the integration by parts formula

$$\int_0^x f(t)(D_q g)t d_q t = [f(t)g(t)]_{t=0}^{t=x} - \int_0^x (D_q f)(t)g(qt)d_q t,$$

and (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \tag{1}$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)}, \tag{2}$$

$${}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q(t-qs)^{(\alpha-1)}. \tag{3}$$

Remark 2.1. We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$. To see this, assume that $a \le b \le t$. Then, it is intended to show that

$$t^\alpha \prod_{n=0}^{\infty} \frac{t - aq^n}{t - aq^{\alpha+n}} \ge t^\alpha \prod_{n=0}^{\infty} \frac{t - bq^n}{t - bq^{\alpha+n}}. \tag{4}$$

Let $n \in \mathbb{N}_0$. We show that

$$(t - aq^n)(t - bq^{\alpha+n}) \ge (t - bq^n)(t - aq^{\alpha+n}). \tag{5}$$

Indeed, expanding both sides of the inequality (5) we obtain

$$\begin{aligned}
&t^2 - tbq^{\alpha+n} - taq^n + aq^n bq^{\alpha+n} \ge t^2 - taq^{\alpha+n} - tbq^n + bq^n aq^{\alpha+n} \\
\Leftrightarrow &q^n(aq^\alpha + b) \ge q^n(bq^\alpha + a) \\
\Leftrightarrow &b - a \ge q^\alpha(b - a) \\
\Leftrightarrow &1 \ge q^\alpha.
\end{aligned}$$

Since inequality (5) implies inequality (4) we are done with the proof.

The following definition was considered first in [1]

Definition 2.2. Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0,1].$$

The fractional q-derivative of order $\alpha \ge 0$ is defined by $(D_q^0 f)(x) = f(x)$ and $(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x)$ for $\alpha > 0$, where m is the smallest integer greater or equal than α.

Let us now list some properties that are already known in the literature. Its proof can be found in [1, 15].

Lemma 2.3. *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:*

1. $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,
2. $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

The next result is important in the sequel. Since we didn't find it in the literature we provide a proof here.

Theorem 2.4. *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^k f)(0). \tag{6}$$

Proof. Let α be any positive number. We will do a proof using induction on p.

Suppose that $p = 1$. Using formula (3) we get:

$${}_tD_q[(x-t)^{(\alpha-1)}f(t)] = (x-qt)^{(\alpha-1)}{}_tD_q f(t) - [\alpha-1]_q(x-qt)^{(\alpha-2)}f(t).$$

Therefore,

$$\begin{aligned}(I_q^\alpha D_q f)(x) &= \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)}(D_q f)(t)d_qt \\ &= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-2)}f(t)d_qt + \frac{1}{\Gamma_q(\alpha)}[(x-t)^{(\alpha-1)}f(t)]_{t=0}^{t=x} \\ &= (D_q I_q^\alpha f)(x) - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}f(0).\end{aligned}$$

Suppose now that (6) holds for $p \in \mathbb{N}$. Then,

$$\begin{aligned}(I_q^\alpha D_q^{p+1} f)(x) &= (I_q^\alpha D_q^p D_q f)(x) \\ &= (D_q^p I_q^\alpha D_q f)(x) - \sum_{k=0}^{p-1}\frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^{k+1}f)(0) \\ &= D_q^p\left[(D_q I_q^\alpha f)(x) - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}f(0)\right] - \sum_{k=0}^{p-1}\frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^{k+1}f)(0) \\ &= (D_q^{p+1}I_q^\alpha f)(x) - \frac{x^{\alpha-1-p}}{\Gamma_q(\alpha-p)}f(0) - \sum_{k=1}^{p}\frac{x^{\alpha-(p+1)+k}}{\Gamma_q(\alpha+k-(p+1)+1)}(D_q^k f)(0) \\ &= (D_q^{p+1}I_q^\alpha f)(x) - \sum_{k=0}^{p}\frac{x^{\alpha-(p+1)+k}}{\Gamma_q(\alpha+k-(p+1)+1)}(D_q^k f)(0).\end{aligned}$$

The theorem is proved. □

3 Fractional boundary value problem

We shall consider now the question of existence of nontrivial solutions to the following problem:

$$(D_q^\alpha y)(x) = -f(x, y(x)), \quad 0 < x < 1, \tag{7}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 0, \tag{8}$$

where $1 < \alpha \leq 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function (this is the q-analogue of the fractional differential problem considered in [5]). To that end we need the following theorem (see [8, 12]).

Theorem 3.1. *Let $\mathcal{B}$ be a Banach space, and let $C \subset \mathcal{B}$ be a cone. Assume Ω_1, Ω_2 are open disks contained in $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a completely continuous operator such that*

$$\|Ty\| \geq \|y\|, \ y \in C \cap \partial\Omega_1 \ \text{and} \ \|Ty\| \leq \|y\|, \ y \in C \cap \partial\Omega_2.$$

Then T has at least one fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let us put $p = 2$. In view of item 2 of Lemma 2.3 and Theorem 2.4 we see that

$$\begin{aligned}(D_q^\alpha y)(x) = -f(x, y(x)) &\Leftrightarrow (I_q^\alpha D_q^2 I_q^{2-\alpha} y)(x) = -I_q^\alpha f(x, y(x)) \\ &\Leftrightarrow y(x) = c_1 x^{\alpha-1} + c_2 x^{\alpha-2} - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t, y(t)) d_q t,\end{aligned}$$

for some constants $c_1, c_2 \in \mathbb{R}$. Using the boundary conditions given in (8) we take $c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qt)^{(\alpha-1)} f(t, y(t)) d_q t$ and $c_2 = 0$ to get

$$\begin{aligned}y(x) &= \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1 - qt)^{(\alpha-1)} x^{\alpha-1} f(t, y(t)) d_q t \\ &\qquad - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t, y(t)) d_q t \\ &= \frac{1}{\Gamma_q(\alpha)} \left[\int_0^x \left([x(1 - qt)]^{(\alpha-1)} - (x - qt)^{(\alpha-1)} \right) f(t, y(t)) d_q t \right. \\ &\qquad \left. + \int_x^1 [x(1 - qt)]^{(\alpha-1)} f(t, y(t)) d_q t \right].\end{aligned}$$

If we define a function G by

$$G(x,t)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}(x(1-t))^{(\alpha-1)}-(x-t)^{(\alpha-1)}, & 0\le t\le x\le 1,\\ (x(1-t))^{(\alpha-1)}, & 0\le x\le t\le 1,\end{cases}$$

then, the following result follows.

Lemma 3.2. *y is a solution of the boundary value problem* (7)-(8) *if, and only if, y satisfies the integral equation*

$$y(x)=\int_0^1 G(x,qt)f(t,y(t))d_qt.$$

Remark 3.3. If we let $\alpha=2$ in the function G, then we get a particular case of the Green function obtained in [16], namely,

$$G(x,t)=\begin{cases}t(1-x), & 0\le t\le x\le 1\\ x(1-t), & 0\le x\le t\le 1.\end{cases}$$

Some properties of the function G needed in the sequel are now stated and proved.

Lemma 3.4. *Function G defined above satisfies the following conditions:*

$$G(x,qt)\ge 0 \text{ and } G(x,qt)\le G(qt,qt) \text{ for all } 0\le x,t\le 1. \tag{9}$$

Proof. We start by defining two functions $g_1(x,t)=(x(1-t))^{(\alpha-1)}-(x-t)^{(\alpha-1)}$, $0\le t\le x\le 1$ and $g_2(x,t)=(x(1-t))^{(\alpha-1)}$, $0\le x\le t\le 1$. It is clear that $g_2(x,qt)\ge 0$. Now, in view of Remark 2.1 we get,

$$\begin{aligned}g_1(x,qt)&=x^{\alpha-1}(1-qt)^{(\alpha-1)}-x^{\alpha-1}(1-q\frac{t}{x})^{(\alpha-1)}\\ &\ge x^{\alpha-1}(1-qt)^{(\alpha-1)}-x^{\alpha-1}(1-qt)^{(\alpha-1)}=0.\end{aligned}$$

Moreover, for $t\in(0,1]$ we have that

$$\begin{aligned}{}_xD_q g_1(x,t)&={}_xD_q[(x(1-t))^{(\alpha-1)}-(x-t)^{(\alpha-1)}]\\ &=[\alpha-1]_q(1-t)^{(\alpha-1)}x^{\alpha-2}-[\alpha-1]_q(x-t)^{(\alpha-2)}\\ &=[\alpha-1]_q x^{\alpha-2}\left[(1-t)^{(\alpha-1)}-\left(1-\frac{t}{x}\right)^{(\alpha-2)}\right]\\ &\le[\alpha-1]_q x^{\alpha-2}\left[(1-t)^{(\alpha-1)}-(1-t)^{(\alpha-2)}\right]\\ &\le 0,\end{aligned}$$

which implies that $g_1(x,t)$ is decreasing with respect to x for all $t \in (0,1]$. Therefore,

$$g_1(x,qt) \leq g_1(qt,qt), \quad 0 < x,t \leq 1. \tag{10}$$

Now note that $G(0,qt) = 0 \leq G(qt,qt)$ for all $t \in [0,1]$. Therefore, by (10) and the definition of g_2 (it is obviously increasing in x) we conclude that $G(x,qt) \leq G(qt,qt)$ for all $0 \leq x,t \leq 1$. This finishes the proof. □

Let $\mathcal{B} = C[0,1]$ be the Banach space endowed with norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Define the cone $C \subset \mathcal{B}$ by

$$C = \{u \in \mathcal{B} : u(t) \geq 0\}.$$

Remark 3.5. It follows from the nonnegativeness and continuity of G and f that the operator $T : C \to \mathcal{B}$ defined by

$$(Tu)(x) = \int_0^1 G(x,qt)f(t,u(t))d_qt,$$

satisfies $T(C) \subset C$ and is completely continuous.

For our purposes, let us define two constants

$$M = \left(\int_0^1 G(qt,qt)d_qt\right)^{-1}, \quad N = \left(\int_{\tau_1}^{\tau_2} G\left(qt,qt\right)d_qt\right)^{-1},$$

where $\tau_1 \in \{0, q^m\}$ and $\tau_2 = q^n$ with $m,n \in \mathbb{N}_0$, $m > n$. Our existence result is now given.

Theorem 3.6. *Let $f(t,u)$ be a nonnegative continuous function on $[0,1] \times [0,\infty)$. If there exists two positive constants $r_2 > r_1 > 0$ such that*

$$f(t,u) \leq Mr_2, \ \textit{for}\ (t,u) \in [0,1] \times [0,r_2], \tag{11}$$
$$f(t,u) \geq Nr_1, \ \textit{for}\ (t,u) \in [\tau_1,\tau_2] \times [0,r_1], \tag{12}$$

then problem (7)-(8) has a solution y satisfying $r_1 \leq \|y\| \leq r_2$.

Proof. Since the operator $T : C \to C$ is completely continuous we only have to show that the operator equation $y = Ty$ has a solution satisfying $r_1 \leq \|y\| \leq r_2$.

Let $\Omega_1 = \{y \in C : \|y\| < r_1\}$. For $y \in C \cap \partial\Omega_1$, we have $0 \le y(t) \le r_1$ on $[0,1]$. Using (9) and (12), and the definitions of τ_1 and τ_2, we obtain (see page 282 in [7]),

$$\|Ty\| = \max_{0\le x\le 1} \int_0^1 G(x, qt) f(t, y(t)) d_q t \ge N r_1 \int_{\tau_1}^{\tau_2} G(qt, qt)\, d_q t = \|y\|.$$

Let $\Omega_2 = \{y \in C : \|y\| < r_2\}$. For $y \in C \cap \partial\Omega_2$, we have $0 \le y(t) \le r_2$ on $[0,1]$. Using (9) and (11) we obtain,

$$\|Ty\| = \max_{0\le x\le 1} \int_0^1 G(x, qt) f(t, y(t)) d_q t \le M r_2 \int_0^1 G(qt, qt)\, d_q t = \|y\|.$$

Now an application of Theorem 3.1 concludes the proof. $\square$

References

[1] R. P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc. **66** (1969), 365–370.

[2] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Acad. Publ., Dordrecht, 1999.

[3] W. A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinburgh Math. Soc. (2) **15** (1966/1967), 135–140.

[4] F. M. Atici and P. W. Eloe, Fractional q-calculus on a time scale, J. Nonlinear Math. Phys. **14** (2007), no. 3, 333–344.

[5] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. **311** (2005), no. 2, 495–505.

[6] Ernst T, The History of q-Calculus and a New Method, U. U. D. M. Report 2000:16, ISSN 1101-3591, Department of Mathematics, Uppsala University, 2000.

[7] H. Gauchman, Integral inequalities in q-calculus, Comput. Math. Appl. **47** (2004), no. 2-3, 281–300.

[8] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, MA, 1988.

[9] Jackson F.H.: On q-functions and a certain difference operator. Trans. Roy Soc. Edin. 46, (1908), 253-281.

[10] Jackson F.H.: On q-definite integrals. Quart. J. Pure and Appl. Math. 41 (1910), 193-203.

[11] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[12] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron, P. Noordhoff Ltd. Groningen, 1964.

[13] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[15] P. M. Rajković, S. D. Marinković and M. S. Stanković, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math. **1** (2007), no. 1, 311–323.

[16] M. El-Shahed and H. A. Hassan, Positive solutions of q-difference equation, Proc. Amer. Math. Soc. **138** (2010), no. 5, 1733–1738.

(Received August 4, 2010)